J. Fluid Mech. (2009), *vol.* 624, *pp.* 339–360. © 2009 Cambridge University Press
doi:10.1017/S0022112008005508 Printed in the United Kingdom

Spectral data for travelling water waves: singularities and stability

DAVID P. NICHOLLS†

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, USA

(Received 29 February 2008 and in revised form 2 December 2008)

In this paper we take up the question of the spectral stability of travelling water waves from a new point of view, namely that the spectral data of the water-wave operator linearized about fully nonlinear Stokes waves is analytic as a function of a height parameter. This observation was recently made rigorous by the author using a boundary perturbation approach which is amenable to approximation by a stable high-order numerical method. Using this algorithm, we investigate, for both super- and sub-harmonic disturbances, the evolution of the spectrum, in particular the 'first collision' of eigenvalues and the 'smallest singularity' in the perturbation expansion. The former is studied in response to MacKay & Saffman's (1986) work on the water-wave problem which demonstrated that instability can only arise after the collision of two eigenvalues of opposite Krein signature. However, we present results which show, quite explicitly, that eigenvalue collision (even of opposite Krein signature) is insufficient to conclude instability. With this in mind, we have identified a new criterion for the loss of spectral stability, namely the appearance of a singularity in the expansion of the spectral data (as a function of the height parameter mentioned above). We give some heuristic reasons why this should be so, and then provide complete numerical spectral stability results for four representative depths, two above ($h = \infty, 2$) and two below ($h = 1, 1/2$) Benjamin's (1967) critical value, $h_c \approx 1.363$, above which the Benjamin–Feir instability emerges. We find that the strongest (two-dimensional) instability appears to be among the long waves, but we notice that there is a sharp difference between 'shallow-water' and 'deep-water' waves in that first eigenvalue collision and smallest expansion singularity are synonymous for shallow water, while this is not so in deep water where 'windows of stability' beyond the first eigenvalue collision exist.

1. Introduction

One of the central unresolved questions in the study of free-surface ideal fluid mechanics (the water-wave problem) is a comprehensive dynamic stability theory for travelling wave solutions on a fluid of arbitrary depth. The rigorous justification of the existence of travelling wave solutions to the water-wave equations began with the work of Levi-Civita (1925) and Struik (1926) who proved the existence of two-dimensional (one-dimensional surface) travelling solutions, in the absence of capillarity, in deep and finite-depth water, respectively. Since this time, numerous contributions have been made extending these results to gravity–capillary waves and

† Email address for correspondence: nicholls@math.uic.edu

fully three-dimensional waveforms; we refer the interested reader to the excellent survey articles of Dias & Kharif (1999) and Groves (2007) for a full description of these.

Regarding the dynamic stability of travelling water waves, there is a long and distinguished history dating to the fundamental work of Benjamin & Feir (1967) and Zakharov (1968) in the 1960s which led to the discovery of the Benjamin–Feir instability (see § 2.7) and the Hamiltonian formulation of water waves, to name just two ground-breaking results. Again, the survey article of Dias & Kharif (1999) is an excellent guide to the long list of results in this direction which has typically focused upon either small-amplitude long-waves in shallow water (where the Korteweg–de Vries equation and its generalizations govern) or small waves in deep water (where the Nonlinear Schrödinger equation and its relatives are applicable).

Regarding the *full* Euler equations, much less is known and most results have been limited to numerical simulation. In particular, Longuet-Higgins explored the spectral stability of travelling two-dimensional periodic wavetrains (the Stokes waves) in deep water subject to two-dimensional super-harmonic (see Longuet-Higgins 1978*a*) and sub-harmonic (see Longuet-Higgins 1978*b*) disturbances, respectively. In these, he not only saw the Benjamin–Feir instability in the full equations (a fact which has been subsequently rigorously justified by Bridges & Mielke 1995), but also found the 'bubbles of instability' which are the signature of many systems with Hamiltonian structure. The spectral stability analysis which Longuet-Higgins carried out sought to study the time evolution (in a reference frame moving uniformly with speed c) of the quantity

$$u(x,t) = \bar{u}(x) + \delta e^{\lambda t} v(x), \qquad \delta \ll 1,$$

where $\bar{u}$ is the Stokes wave, v is an envelope and λ is the spectral parameter ($\mathrm{Re}\{\lambda\} > 0$ indicates instability). Insertion of this form into the full Euler equations of free-surface ideal fluid mechanics leads, to leading order, to an eigenproblem for the pair $(\lambda, v(x))$

$$\mathscr{L}[v] = \lambda v.$$

The final specification which must be made is the boundary conditions on the envelope v. If v has the same periodicity as $\bar{u}$, this constitutes a super-harmonic disturbance, otherwise it is sub-harmonic (which, as we will see, can be conveniently specified by Bloch-periodic boundary conditions, see § 2.3). It is easy to see (§ 2.5) that the trivial, 'flat water', solution is neutrally stable ($\mathrm{Re}\{\lambda\} = 0$), so that the interesting stability question becomes the values of λ, in particular their real parts, for non-zero surface deformations.

Longuet-Higgins' approach to this problem was, given a particular choice of $\bar{u}$, to approximate the action of $\mathscr{L}$ on a finite-dimensional subspace of periodic functions, giving a matrix M, that was then passed to a numerical eigensolver which produced eigenvalues and eigenvectors. While rather computationally expensive (M is recomputed for every $\bar{u}$), Longuet-Higgins produced highly accurate approximations of the eigenvalues as the (non-dimensionalized) amplitude, ak, of the Stokes wave was increased to nearly the critical value. In this work, Longuet-Higgins verified the existence of the Benjamin–Feir instability with his sub-harmonic calculations, and showed that this dominates for small ak. He also found that for ak much larger (nearly to the critical amplitude), there is a super-harmonic instability which is much stronger. These results were subsequently generalized by McLean and collaborators (see McLean *et al.* 1981; McLean 1982*a*,*b*) to include the possibility of *three*-dimensional instabilities. In this work, it was shown that, while Benjamin–Feir dominates for small amplitude, for most moderate amplitudes the Stokes waves are primarily unstable to three-dimensional perturbations. This work was continued all the way to the critical wave of greatest amplitude by Kharif (1987) and Kharif & Ramamonjiarisoa (1988) who showed that two-dimensional instabilities eventually become dominant for very nonlinear waves. Further work in this same vein for finite depth has also been carried out (see e.g. Francius & Kharif 2003, 2006).

In the midst of these numerical simulations of the 1980s, MacKay & Saffman (1986) made a fundamental theoretical contribution to the resolution of the stability question. Using the Hamiltonian structure of the water-wave equations found by Zakharov (1968), they showed that spectral instability ($\mathrm{Re}\{\lambda\} > 0$ for some $\lambda \in \mathrm{spec}(\mathscr{L})$) could only occur *after* a collision of eigenvalues on the imaginary axis. Furthermore, this collision must be between two eigenvalues of opposite Krein signature. While very revealing, these results are not conclusive as collision with opposite signature is not *sufficient* to establish instability; please see § 2.8 for further discussion of these results.

In a series of recent papers, the author (partially in collaboration with F. Reitich) has advanced the state-of-the-art in the stable and high-order computation of travelling water waves and their spectral stability. The goal of the research described in this paper is to use these novel computational methodologies to advance the understanding of the stability of travelling water waves, and propose a new criterion for the onset of instability. To begin, Nicholls & Reitich (2006) devised a reliable and highly accurate algorithm for the computation of travelling water waves based upon boundary perturbations (BP). These computations were inspired by their theoretical work (see Nicholls & Reitich 2005) which showed that the BP recursions define *strongly* convergent Taylor series (see (2.9)) for the problem unknowns (velocity potential, free-surface shape and velocity). This BP method was demonstrated to be stable and highly accurate, delivering physical quantities of interest (e.g. frequency, energy density) to 13–14 digits of accuracy (see table 1 of Nicholls & Reitich 2006). The travelling waveforms we use in the present analysis are generated by this very BP approach.

Subsequently, the author (see Nicholls 2007*b*) has derived a BP algorithm for the computation of the spectrum of the water-wave operator linearized about travelling wave solutions. In this paper, the author proved that the spectral data, (λ, v), can also be represented by strongly convergent Taylor series (see (2.10)), and investigated the convergence of a BP algorithm to compute this spectral data. Once again, the scheme was shown to be highly stable and accurate, and the author presented some *preliminary* results on the crossing of eigenvalues. In the current work, we now take these high-fidelity numerical algorithms as proven tools, and use them, for the first time, to truly investigate the stability properties of travelling water waves. Here, we present only two-dimensional results (Stokes waves subject to two-dimensional perturbations), but in principle three-dimensional waveforms (e.g. short-crested waves) subject to three-dimensional perturbations could be addressed. We leave this for future work.

In the present paper, we have found that the *full* spectrum of the linearized water-wave problem can be quickly and accurately simulated for the *complete* family of Stokes waves of a given fundamental period. Using this method, we can rapidly locate the ‘first eigenvalue crossing’ which MacKay & Saffman (1986) showed is a necessary condition for spectral instability. However, we show that this first crossing does *not* necessarily lead to instability as the eigenvalues may simply pass over one another on the imaginary axis. While Longuet-Higgins (1978*b*) displays results of eigenvalue crossing which do not give rise to a bubble of instability, this is only subsequent to other instabilities which have already arisen. Thus, we regard this instance of eigenvalue crossing to be a novel result. Furthermore, according to the formula of MacKay & Saffman (1986), we have found eigenvalue crossing of *opposite* signature which does not lead to instability, a result which we believe is absent from the literature and, thus, also quite novel.

In light of these findings, we propose a new criterion of instability independent of eigenvalue crossing which accommodates the situation where 'first crossing' does lead to instability. Our new criterion (Conjecture 3.2) is based upon observation that our method 'fails' for sufficiently large choices of the perturbation parameter: the expansions for the spectra λ no longer converge. Our numerics reveal that this occurs due to a singularity of the expansion on the *real* axis, and we now conjecture that encountering this singularity and the onset of instability are one and the same. Using the method of Padé summation, we are able to approximate the first such 'expansion singularity' for a fine sampling of the Bloch periods (thereby constituting a full investigation of super- and sub-harmonic disturbances). We do this for four values of the depth, two above ($h=\infty, 2$) and two below ($h=1, 1/2$) the critical value $h_c \approx 1.363$ Benjamin (1967) identified for the onset of the Benjamin–Feir instability. From these studies we show that, in all cases, the strongest instability appears to be in the long-wave regime which, for deep water at least, can be interpreted as the Benjamin–Feir instability. However, there *is* a discrepancy in the behaviour of the spectrum as a function of the perturbation parameter as h increases past h_c. In short, for $h < h_c$ it appears that first eigenvalue crossing and smallest expansion singularity are synonymous so that MacKay and Saffman's criterion suffices. However, for $h > h_c$ we notice that there are significant 'windows of stability' between the first eigenvalue crossing and the smallest expansion singularity, even permitting collision of eigenvalues of opposite signature before the onset of instability. These windows of stability appear to be absent from the current stability literature for water waves, and we view this as a particularly important contribution to the theory.

The rest of the paper is organized as follows. In § 2, we review the equations governing the motion of the surface of an ideal fluid. With this, we present in § 2.1 a change of variables which we have found very useful for both analysis and numerical simulation of water waves. In § 2.2, we review the considerations necessary for the spectral stability analysis of travelling waveforms, and in § 2.3 we discuss Bloch theory in guiding our choice of boundary conditions. In § 2.4, we recall the BP recursions which Nicholls & Reitich (2006) and Nicholls (2007*b*) utilized with great effect in the study of Stokes waves. In § 2.5, we recollect the classical stability calculation for trivial, 'flat water' solutions, and in § 2.6 we recall resonances in this problem which are currently outside the scope of our algorithm. For completeness, we review Benjamin's calculations which display the onset of the Benjamin–Feir instability for $h > h_c$ (§ 2.7), and MacKay & Saffman's work on Krein signature of eigenvalues (§ 2.8). In § 3, we present our new numerical results, more specifically for deep water in § 3.2. Based upon these deep-water considerations, we formulate a new criterion for instability in § 3.3, and explore these new ideas for three more depths in § 3.4. Finally, we state our conclusions in § 4.

2. Governing equations

As mentioned above, we shall consider the motion of the free interface above an ideal (inviscid, irrotational, incompressible) fluid under the influence of gravity. Capillarity can be easily included if desired and, as we found in Nicholls & Reitich

(2005, 2006) and Nicholls (2007*a*, *b*), it is *essential* when considering three-dimensional waveforms and perturbations. If the fluid occupies the domain

$$S_{h,\eta} = \{x \in \boldsymbol{R} \mid -h < y < \eta(x,t)\}$$

with mean depth h and free surface $\eta = \eta(x,t)$, the equations of motion are known to be (see Lamb 1993)

$$\Delta\varphi = 0, \quad \text{in } S_{h,\eta,} \tag{2.1a}$$

$$\partial_y \varphi(x,-h) = 0, \tag{2.1b}$$

$$\partial_t \eta + \partial_x \eta \; \partial_x \varphi - \partial_y \varphi = 0, \quad \text{at } y = \eta, \tag{2.1c}$$

$$\partial_t \varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta = 0, \quad \text{at } y = \eta, \tag{2.1d}$$

where $\varphi = \varphi(x,y,t)$ is the velocity potential ($\boldsymbol{v} = \nabla\varphi$), and g is the constant of gravity. Water of infinite depth can be accommodated by replacing (2.1*b*) with

$$\partial_y \varphi \to 0 \quad \text{as } y \to -\infty, \tag{2.1e}$$

and these equations are also supplemented with initial conditions

$$\eta(x,0) = \eta_0(x), \qquad \varphi(x,\eta_0(x),0) = \xi_0(x), \tag{2.1f}$$

where it suffices (by elliptic theory) to specify φ only at the surface. Boundary conditions must also be enforced to guarantee the existence of a unique solution which, for the study of Stokes waves, are periodic boundary conditions with respect to some lattice $\Gamma \subset \boldsymbol{R}$, i.e.

$$\eta(x+\gamma,t) = \eta(x,t), \quad \varphi(x+\gamma,y,t) = \varphi(x,y,t) \quad \forall \gamma \in \Gamma;$$

this lattice generates the conjugate lattice of wavenumbers (see Mielke 1997),

$$\Gamma' := \{k \in \boldsymbol{R} \mid k \cdot \gamma \in (2\pi)\boldsymbol{Z}, \forall \gamma \in \Gamma\}.$$

In this paper we consider non-dimensionalized quantities and, as such, choose $\Gamma = 2\pi\boldsymbol{Z}$ which gives $\Gamma' = \boldsymbol{Z}$.

If we change coordinates to a reference frame translating uniformly with speed $c \in \boldsymbol{R}$ in the x-direction, then the governing equations become (for water of finite depth)

$$\Delta\varphi = 0, \quad \text{in } S_{h,\eta,} \tag{2.2a}$$

$$\partial_y \varphi(x,-h) = 0, \tag{2.2b}$$

$$\partial_t \eta + c\; \partial_x \eta + \partial_x \eta \; \partial_x \varphi - \partial_y \varphi = 0, \quad \text{at } y = \eta, \tag{2.2c}$$

$$\partial_t \varphi + c\; \partial_x \varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta = 0, \quad \text{at } y = \eta, \tag{2.2d}$$

$$\eta(x,0) = \eta_0(x), \tag{2.2e}$$

$$\varphi(x,\eta_0(x),0) = \xi_0(x), \tag{2.2f}$$

again with periodic boundary conditions.

Finally, we can greatly reduce the size of the problem domain, $S_{h,\eta}$, for (2.2) with the introduction of a 'transparent boundary condition' at a hyperplane $y = -a$, $-h \leqslant -a < -|\eta|_{L^\infty}$ (see Nicholls & Reitich 2005, 2006). This boundary condition is not only important for computational considerations (significantly reducing the domain to be discretized), but also permits a uniform statement of the water-wave problem over *all* depths including the infinite-depth case. As we saw in Nicholls & Reitich (2005, 2006) and Nicholls (2007*b*), we can equivalently state (2.2) (for arbitrary depth) as

$$\Delta\varphi = 0, \quad \text{in } S_{a,\eta,} \tag{2.3a}$$

$$\partial_y\varphi - T[\varphi] = 0, \quad \text{at } y = -a, \tag{2.3b}$$

$$\partial_t\eta + c\ \partial_x\eta + \partial_x\eta\ \partial_x\varphi - \partial_y\varphi = 0, \quad \text{at } y = \eta, \tag{2.3c}$$

$$\partial_t\varphi + c\ \partial_x\varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta = 0, \quad \text{at } y = \eta, \tag{2.3d}$$

$$\eta(x, 0) = \eta_0(x), \tag{2.3e}$$

$$\varphi(x, \eta_0(x), 0) = \xi_0(x), \tag{2.3f}$$

where

$$T(a)[\psi] = \sum_{k\in\Gamma'} |k| \tanh((h-a)|k|)\hat{\psi}_k e^{ikx},$$

and $\hat{\psi}_k$ is the k-th Fourier coefficient of $\psi(x)$. If we define $D := -i\partial_x$, then the operator T can be written as the Fourier multiplier

$$T[\psi] = |D| \tanh((h-a)|D|)[\psi].$$

In the case of infinite depth, we have $T = |D|$.

2.1. *Change of variables*

We have found the change of variables

$$x' = x, \qquad y' = a\left(\frac{y - \eta(x,t)}{a + \eta(x,t)}\right), \qquad t' = t, \tag{2.4}$$

very useful in the rigorous analysis (see Nicholls & Reitich 2001*a*, 2003, 2004*a*, 2005; Nicholls 2007*b*) and numerical simulation (see Nicholls & Reitich 2001*b*, 2004*b*) of water waves in two and three dimensions. We note that these coordinates are known as σ coordinates in the atmospheric science community (see Phillips 1957) and the C method in electromagnetics (see Chandezon, Maystre & Raoult 1980; Li *et al.* 1999). As we have already shown (see Nicholls & Reitich 2005, 2006), using this change of variables and setting

$$u(x', y', t') := \varphi(x', (a+\eta)y'/a + \eta, t')$$

results in (dropping primes) the transformation of (2.3) to

$$\Delta u(x, y) = F(x, y; \eta, u), \quad \text{in } S_{a,0,} \tag{2.5a}$$

$$\partial_y u(x, -a) - T[u(x, -a)] = J(x; \eta, u), \tag{2.5b}$$

$$\partial_t\eta(x, t) + c\ \partial_x\eta(x, t) - \partial_y u(x, 0, t) = Q(x; \eta, u), \tag{2.5c}$$

$$\partial_t u(x, 0, t) + c\ \partial_x u(x, 0, t) + g\eta(x, t) = R(x; \eta, u), \tag{2.5d}$$

where the right-hand sides are explicitly given in Nicholls & Reitich (2005, 2006) and share the important feature of being $\mathcal{O}(\eta)$. These are supplemented with the initial conditions (2.3*e*) and (2.3*f*), and periodic boundary conditions each of which is trivially transformed.

2.2. *Spectral stability analysis*

The spectral stability analysis that we have in mind is fully described in Nicholls (2007*b*), but we briefly outline it here for completeness. To begin, consider a travelling wave solution of (2.1), i.e. a *steady* solution

$$(\bar{u}, \bar{\eta}, \bar{c}) = (\bar{u}(x, y), \bar{\eta}(x), \bar{c})$$

of (2.5). With this, we seek solutions to the *full* problem (2.5) in a frame travelling with speed $c = \bar{c}$ of the form

$$u(x, y, t) = \bar{u}(x, y) + \delta\tilde{u}(x, y, t), \quad \eta(x, t) = \bar{\eta}(x) + \delta\tilde{\eta}(x, t),$$

where $\delta \ll 1$ measures the magnitude of the small perturbation of the travelling state. Inserting this into (2.5) we find, to order $\mathscr{O}(\delta)$, that the perturbations $\tilde{u}$ and $\tilde{\eta}$ satisfy

$$\Delta\tilde{u} = F_u(\bar{u}, \bar{\eta})[\tilde{u}] + F_\eta(\bar{u}, \bar{\eta})[\tilde{\eta}], \quad \text{in } S_{a,0,} \tag{2.6a}$$

$$\partial_y\tilde{u}(x, -a) - T[\tilde{u}(x, -a)] = J_u(\bar{u}, \bar{\eta})[\tilde{u}] + J_\eta(\bar{u}, \bar{\eta})[\tilde{\eta}], \tag{2.6b}$$

$$\partial_t\tilde{\eta}(x, t) + \bar{c}\ \partial_x\tilde{\eta}(x, t) - \partial_y\tilde{u}(x, 0, t) = Q_u(\bar{u}, \bar{\eta})[\tilde{u}] + Q_\eta(\bar{u}, \bar{\eta})[\tilde{\eta}], \tag{2.6c}$$

$$\partial_t\tilde{u}(x, 0, t) + \bar{c}\ \partial_x\tilde{u}(x, 0, t) + g\tilde{\eta}(x, t) = R_u(\bar{u}, \bar{\eta})[\tilde{u}] + R_\eta(\bar{u}, \bar{\eta})[\tilde{\eta}], \tag{2.6d}$$

where u and η variations are denoted by subscripts; the precise forms for the right-hand side are, again, recorded in Nicholls (2007*b*).

We now posit 'spectral stability' forms for $\tilde{u}$ and $\tilde{\eta}$

$$\tilde{u}(x, y, t) = e^{\lambda t} v(x, y), \quad \tilde{\eta}(x, t) = e^{\lambda t}\zeta(x),$$

so that the values of λ determine the *spectral* stability of the travelling waves $(\bar{u}, \bar{\eta})$:

(*a*) $\mathrm{Re}\{\lambda\} > 0$ for *any* λ implies spectral instability,
(*b*) $\mathrm{Re}\{\lambda\} \leqslant 0$ for *all* λ implies weak spectral stability,
(*c*) $\mathrm{Re}\{\lambda\} < 0$ for *all* λ implies strong spectral stability.

Inserting these into (2.6), we find that v and ζ satisfy

$$\Delta v = F_u(\bar{u}, \bar{\eta})[v] + F_\eta(\bar{u}, \bar{\eta})[\zeta], \quad \text{in } S_{a,0,} \tag{2.7a}$$

$$\partial_y v(x, -a) - T[v(x, -a)] = J_u(\bar{u}, \bar{\eta})[v] + J_\eta(\bar{u}, \bar{\eta})[\zeta], \tag{2.7b}$$

$$[\lambda + \bar{c}\ \partial_x]\,\zeta(x) - \partial_y v(x, 0) = Q_u(\bar{u}, \bar{\eta}, \lambda)[v] + Q_\eta(\bar{u}, \bar{\eta}, \lambda)[\zeta], \tag{2.7c}$$

$$[\lambda + \bar{c}\ \partial_x]\, v(x, 0) + g\ \zeta(x) = R_u(\bar{u}, \bar{\eta}, \lambda)[v] + R_\eta(\bar{u}, \bar{\eta}, \lambda)[\zeta]. \tag{2.7d}$$

2.3. *Bloch theory*

The final specification we make for our spectral stability problem (2.7), which we now write abstractly as

$$\mathscr{A}(x, y)[(v, \zeta)] = \lambda(v, \zeta), \tag{2.8}$$

is the boundary conditions which v and ζ must satisfy. For this, we use the 'Generalized Principle of Reduced Instability' (essentially, Floquet theory, see Deconinck & Kutz 2006) developed by Mielke (1997), inspired by the Bloch theory of Schrödinger equations with periodic potentials (see Reed & Simon 1978). This method allows perturbations

$$(v, \zeta) \in H^2(\boldsymbol{R} \times [-a, 0]) \times H^2(\boldsymbol{R}),$$

or even

$$(v, \zeta) \in H^2_{lu}(\boldsymbol{R} \times [-a, 0]) \times H^2_{lu}(\boldsymbol{R}),$$

which lie in the Sobolev spaces built upon the class of uniformly local L^2 functions (see Mielke 1997). Mielke's method reduces this general setting to the study of the 'Bloch waves', e.g.

$$\zeta(x) = \mathrm{e}^{\mathrm{i}p\cdot x} Z(x),$$

where $Z \in H^2(\boldsymbol{R}/\Gamma)$ and Γ is the lattice of periodicity for the linear operator $\mathscr{A} = \mathscr{A}(\bar{u}(x,\cdot), \bar{\eta}(x), \bar{c})$ in the x variable.

Since $Z \in H^2(\boldsymbol{R}/\Gamma)$, it suffices to consider $p \in P(\Gamma')$, the fundamental cell of wavenumbers (e.g. if $\Gamma = (2\pi)\boldsymbol{Z}$, then $\Gamma' = \boldsymbol{Z}$, and $P(\Gamma') = [0, 1]$). Thus, (2.8) delivers the spectral problem (see Mielke 1997)

$$\mathscr{A}_p[(V, Z)] = \lambda(V, Z),$$

where $\mathscr{A}_p$ is the 'Bloch operator'

$$\mathscr{A}_p[(V, Z)] := \mathrm{e}^{-\mathrm{i}p\cdot x}\mathscr{A}[\mathrm{e}^{\mathrm{i}p\cdot x}(V, Z)].$$

The crucial spectral identity (see Mielke 1997, Theorems 2.1 and A.4) is

$$L^2\text{–spec}(\mathscr{A}) = L^2_{lu}\text{–spec}(\mathscr{A}) = \text{closure}\left(\bigcup_{p\in P(\Gamma')} \text{spec}(\mathscr{A}_p)\right).$$

Thus, we can obtain information about stability with respect to *all* of these perturbations by simply considering

$$(V, Z) \in H^2((\boldsymbol{R}/\Gamma) \times [-a, 0]) \times H^2(\boldsymbol{R}/\Gamma)$$

with $p \in P(\Gamma')$ appearing as a parameter (see Mielke 1997).

As remarked in Nicholls (2007*b*), this analysis is equivalent to requiring that the functions $v(x, y)$ and $\zeta(x)$ satisfy the 'Bloch-boundary conditions'

$$v(x+\gamma, y) = \mathrm{e}^{\mathrm{i}p\gamma} v(x, y), \quad \zeta(x+\gamma) = \mathrm{e}^{\mathrm{i}p\gamma}\zeta(x), \quad \forall \gamma \in \Gamma.$$

Notice that if p is a rational number, then these functions will be periodic with respect to the lattice Γ. Such functions can be expanded as

$$\zeta(x) = \sum_{k\in\Gamma'} \hat{\zeta}_k \mathrm{e}^{\mathrm{i}(k+p)x}, \quad v(x, y) = \sum_{k\in\Gamma'} \hat{v}_k(y)\mathrm{e}^{\mathrm{i}(k+p)x},$$

(see McLean *et al.* 1981; McLean 1982*a*,*b*) which is important in our numerical implementation.

2.4. *Transformed field expansions*

In recent work (see Nicholls & Reitich 2005), we showed that steady solutions of (2.5) exist in branches dependent on a perturbation parameter $\varepsilon \in \boldsymbol{R}$, giving rise to the *strongly* convergent expansions

$$\bar{u}(x, y, \varepsilon) = \sum_{n=1}^{\infty} \bar{u}_n(x, y)\varepsilon^n, \quad \bar{\eta}(x, \varepsilon) = \sum_{n=1}^{\infty} \bar{\eta}_n(x)\varepsilon^n, \quad \bar{c}(\varepsilon) = \sum_{n=0}^{\infty} \bar{c}_n\varepsilon^n. \tag{2.9}$$

We have subsequently shown (see Nicholls 2007*b*) that the linearized quantities (v, ζ, λ) also depend (strongly) analytically upon ε and can similarly be expanded as

$$v(x, y, \varepsilon) = \sum_{n=0}^{\infty} v_n(x, y)\varepsilon^n, \quad \zeta(x, \varepsilon) = \sum_{n=0}^{\infty} \zeta_n(x)\varepsilon^n, \quad \lambda(\varepsilon) = \sum_{n=0}^{\infty} \lambda_n\varepsilon^n. \tag{2.10}$$

REMARK 2.1. *To clarify this notion of 'strong convergence', we remark that Theorem 4.2 in Nicholls & Reitich (2005) states that, for some positive constants C and B, and integer $s > 1/2$,*

$$\|\bar{u}_n\|_{H^{s+2}} \leqslant C\frac{B^n}{(n+1)^2}, \quad \|\bar{\eta}_n\|_{H^{s+5/2}} \leqslant C\frac{B^n}{(n+1)^2}, \quad |\bar{c}_{n-1}| \leqslant C\frac{B^{n-1}}{(n+1)^2},$$

where H^s is the Sobolev space of functions with s-many L^2 derivatives. Furthermore, under certain conditions of non-resonance (see § 2.6), Theorem 5.2 in Nicholls (2007b) shows that, for some positive constants K and D,

$$\|v_n\|_{H^{s+2}} \leqslant K\frac{D^n}{(n+1)^2}, \quad \|\zeta_n\|_{H^{s+5/2}} \leqslant K\frac{D^n}{(n+1)^2}, \quad |\lambda_n| \leqslant K\frac{D^n}{(n+1)^2}.$$

The objective of this spectral stability analysis is to 'follow' the eigenvalues $\lambda = \lambda(\varepsilon)$ as ε is varied, particularly any which may move into the positive real half-plane giving rise to a dynamic instability. At this point, we comment that the parameter ε is not uniquely defined and, in fact, two choices at least are widely used. The classical choice first used by Stokes in the 1880s is determined by enforcing the condition that $\bar{\eta}_n$ be orthogonal to $\bar{\eta}_1$ which, computationally, is very natural. However, Schwartz (1974) pointed out that this parameterization introduced an artificial singularity into the expansion causing divergence of the series (2.9) for amplitudes *lower* than that of the critical Stokes wave. He advocated a new parameterization, which we name after him, that is essentially parameterization by waveheight (see Schwartz 1974; Roberts 1983; Marchant & Roberts 1987; Nicholls & Reitich 2006). In this paper, ε will always refer to this Schwartz parameterization.

Inserting the expansions, (2.10), into (2.7), we realize the 'Transformed Field Expansions' (TFE) recursions for the spectral data

$$\Delta v_n = \mathscr{F}_n, \quad \text{in } S_{a,0,} \tag{2.11a}$$

$$\partial_y v_n(x, -a) - T[v_n(x, -a)] = \mathscr{J}_n, \tag{2.11b}$$

$$[\lambda_0 + \bar{c}_0\ \partial_x]\,\zeta_n(x) - \partial_y v_n(x, 0) = \mathscr{Q}_n - \lambda_n \zeta_0(x) - \sum_{l=1}^{n-1} \lambda_{n-l}\zeta_l(x), \tag{2.11c}$$

$$[\lambda_0 + \bar{c}_0\ \partial_x]\,v_n(x, 0) + g\ \zeta_n(x) = \mathscr{R}_n - \lambda_n v_0(x, 0) - \sum_{l=1}^{n-1} \lambda_{n-l} v_l(x, 0), \tag{2.11d}$$

together with Bloch periodicity, where the right-hand sides are given in Nicholls (2007*b*). These TFE recursions not only permit the direct estimation of the quantities $\{v_n, \zeta_n, \lambda_n\}$, but also, as we shall see, lead to a stable high-order algorithm for the numerical study of dynamic stability of travelling water waves.

2.5. *Zeroth order*

Before proceeding to the description of our algorithm, we summarize the discussion given in Nicholls (2007*b*) of solutions to (2.11) at order zero. In this case, the right-hand side of (2.11) is zero, and we find that solutions of (2.11*a*), (2.11*b*) and the Bloch-boundary conditions are

$$v_0(x, y) = \sum_k A_k \frac{\cosh(|k+p|\,(y+h))}{\cosh(|k+p|\,h)} \mathrm{e}^{\mathrm{i}(k+p)x}.$$

The boundary conditions imply that

$$\zeta_0(x) = \sum_k \hat{\zeta}_{0,k} \mathrm{e}^{\mathrm{i}(k+p)x},$$

while (2.11*c*) and (2.11*d*) give

$$M_k w_k = 0 \qquad \forall k \in \Gamma', \tag{2.12}$$

for the matrix/vector pair

$$M_k := \begin{pmatrix} \lambda_0 + \mathrm{i}\bar{c}_0(k+p) & -|k+p| \tanh(|k+p|\, h) \\ g & \lambda_0 + \mathrm{i}\bar{c}_0(k+p) \end{pmatrix}, \quad w_k := \begin{pmatrix} \hat{\zeta}_{0,k} \\ A_k \end{pmatrix}. \tag{2.13}$$

Clearly, (2.12) only delivers *non-trivial* solutions when the matrix M_k is singular for some wavenumber $k \in \Gamma'$; this occurs when the determinant function

$$\Lambda(k, p; \lambda_0, \bar{c}_0, g, h) := (\lambda_0 + \mathrm{i}\bar{c}_0 \cdot (k+p))^2 + g\, |k+p| \tanh(h\, |k+p|), \tag{2.14}$$

is equal to zero. Given that $(p, \bar{c}_0, g, h)$ are *fixed*, for a given $\kappa \in \Gamma'$, we solve for the *unique* (up to sign) $\lambda_0^s(\kappa)$, $s = \pm 1$, such that $\Lambda(\kappa, p; \lambda_0^s, \bar{c}_0, g, h) = 0$, i.e.

$$\lambda_0^s(\kappa) = i \left[-\bar{c}_0 \cdot (\kappa + p) + s\omega_{\kappa+p} \right], \tag{2.15}$$

where

$$\omega_k^2 := g\, |k| \tanh(h\, |k|). \tag{2.16}$$

Since all of the $\lambda_0^s(\kappa)$ are purely imaginary, we recover the classical weak stability result for trivial (flat) travelling water waves. In this case, we find solutions of (2.12) of the form

$$\hat{\zeta}_{0,\kappa} = \tau\, |\kappa + p| \tanh(|k+p|\, h), \quad A_\kappa = \tau(\lambda_0 + \mathrm{i}\bar{c}_0 \cdot (\kappa + p)),$$

for any $\tau \in \mathbf{C}$ which, together with $\lambda_0 = \lambda_0^s$, are the starting point for our algorithm.

2.6. *Resonance*

We now turn to the restriction of our theory, namely that certain 'resonant' configurations must be excluded. To see why this is the case, recall that, to start our algorithm, for a fixed set of parameters $(\Gamma, p, \bar{c}_0, g, h)$ we can select, for every $\kappa \in \Gamma'$, a unique (up to sign) value $\lambda_0^s(\kappa)$, (2.15), such that

$$\Lambda(\kappa, p; \lambda_0^s, \bar{c}_0, g, h) = 0$$

(see (2.14)). As we saw in (Nicholls 2007*b*) if $\Lambda(k, p) \neq 0$ for all $k \neq \kappa$, then (2.11) can be uniquely solved at any order n. Viewing p as the adjustable 'configuration parameter' we define the set of permissible, 'non-resonant' p, for a fixed κ, by

$$\Omega_\kappa(\Gamma) := \{ p \in P(\Gamma') \mid \Lambda(k, p; \lambda_0^s, \bar{c}_0, g, h) \neq 0 \ \forall\ k \in \Gamma' \backslash \{\kappa\} \}. \tag{2.17}$$

As we shall see in § 3, the resonant quasi-periods $p \notin \Omega_\kappa(\Gamma)$ are excluded from our computations. While this is a shortcoming of our approach, we regard it as a minor one. First, this set is not very large (in two dimensions on water of infinite depth, there are only three values ($p = 0, 1/4, 3/4$) in the continuum [0, 1) see Nicholls 2007*b*). Additionally, we have been able to choose values of p quite close to the resonant values and, we believe, gleaned useful information about these resonances. Finally, these resonant values would give rise to particularly strong instabilities which are *independent* of the travelling wave shape (as they appear at perturbation order zero), and thus require a more subtle analysis which we leave to future research. Instead, we focus upon the non-resonant configurations featuring a dependence upon the travelling waveform which we can detect.

2.7. *The Benjamin–Feir instability*

Before turning to our numerical results, we briefly discuss two crucial contributions to the theory of stability of travelling water waves: the Benjamin–Feir instability (see Benjamin 1967; Benjamin & Feir 1967) and the work of MacKay & Saffman (1986) on Krein signature of eigenvalues of the linearized problem. We focus upon the former in this subsection and defer a discussion of the latter until § 2.8. As this is background material and not the original work of the author, we present a brief outline and refer the interested reader to the original papers.

Based upon numerous and careful experimental observations, Benjamin and Feir came to the conclusion that periodic wavetrains in deep water were unstable to 'sidebands' of the wavetrain frequency. To mathematize this, they considered a small amplitude ($a \ll 1$) travelling wave of the form

$$H = \bar{\eta} + a\cos(\zeta) + ka^2 P\cos(2\zeta),$$
$$\Phi = \frac{\omega a}{k}\frac{\cosh(k(y+h))}{\sinh(hk)}\sin(\zeta) + \omega a^2 Q\frac{\cosh(2k(y+h))}{\sinh(2hk)}\sin(2\zeta),$$

(H and Φ are meant to approximate travelling wave solutions η and φ of (2.1)) where

$$\zeta = kx - \omega t, \quad \bar{\eta}, P, Q \in \boldsymbol{R},$$

(see Benjamin 1967) which is the Stokes expansion to order two. They then considered perturbations of this travelling wave of the form

$$\eta = H + \iota\tilde{\eta}, \quad \varphi = \Phi + \iota\tilde{\varphi},$$

$\iota \ll 1$, with the Benjamin–Feir-type perturbation

$$\tilde{\eta} = \tilde{\eta}_1 + \tilde{\eta}_2,$$

where

$$\tilde{\eta}_j = \varepsilon_j\cos(\zeta_j) + ka\{A_j\cos(\zeta+\zeta_j) + B_j\cos(\zeta-\zeta_j)\} + \mathcal{O}(k^2a^2\varepsilon_j).$$

Here

$$\zeta_j = k(1\pm\kappa)x - \omega(1\pm\delta)t - \gamma_j,$$

$\kappa, \delta = \mathcal{O}(ka)$, and

$$\frac{\delta}{\kappa} = \frac{1}{2}(1 + 2kh\mathrm{csch}(2kh)) =: \lambda.$$

In their analysis $\varepsilon_j = \varepsilon_j(t)$, $\gamma_j = \gamma_j(t)$ are slowly varying functions of time to be determined. An appropriate form for $\tilde{\varphi}$ can also be provided (see Benjamin 1967) and, up to order $\mathcal{O}(\omega k^2a^2\varepsilon_j + \omega^2ka^2\varepsilon_j)$ the boundary conditions (2.1*c*) and (2.1*d*), linearized about (H, Φ), imply the evolution equations

$$\frac{\mathrm{d}\varepsilon_{1,2}}{\mathrm{d}t} = \frac{1}{2}\omega k^2a^2X(kh)\sin(\theta(t))\varepsilon_{2,1}, \tag{2.18}$$

where $X(kh)$ is given in (30) of Benjamin (1967), and

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega k^2a^2X(kh)\left\{1 + \frac{\varepsilon_1^2+\varepsilon_2^2}{2\varepsilon_1\varepsilon_2}\cos(\theta)\right\} - \omega\delta^2Y(kh) \tag{2.19}$$

comes from the sum of the evolution equations for $\gamma_j(t)$, and $Y(kh)$ is given in (36) of Benjamin (1967).

Explicit solutions of (2.18) can be written down, and an analysis of (2.19) reveals that $\varepsilon_j(t)$ are periodic and finitely bounded if

$$2k^2a^2X(kh) < \delta^2 Y(kh).$$

However, if

$$2k^2a^2X(kh) \geqslant \delta^2 Y(kh),$$

then the ε_j will grow as $t \to \infty$, with *exponential* growth if strict inequality holds. Since $Y(kh) > 0$ and δ can be chosen arbitrarily small, the matter of stability hinges on the sign of $X(kh)$,

$$X(kh) \geqslant 0 \quad \text{if and only if} \quad kh \geqslant 1.363$$

implies instability. Typically, as we have done in this paper, non-dimensionalization chooses $k = 1$, and thus the critical depth for the onset of the Benjamin–Feir instability is $h_c \approx 1.363$.

2.8. *Krein signature*

We conclude our theoretical discussions with an overview of MacKay & Saffman's (see MacKay & Saffman 1986) results on Krein signature of colliding eigenvalues. These authors focus upon the Hamiltonian formulation of water waves due to Zakharov (1968) which states that (2.1) are equivalent to

$$\partial_t \eta = \delta_\xi H_Z, \quad \partial_t \xi = -\delta_\eta H_Z,$$

where $\xi(x, t) := \varphi(x, \eta(x, t), t)$ is the velocity potential at the free surface, and the Hamiltonian is given by $H_Z = K + V$, the sum of the kinetic and potential energies

$$K = \frac{1}{2}\int\int_{-h}^{\eta} |\nabla\varphi|^2 \, \mathrm{d}y \, \mathrm{d}x, \quad V = \int \frac{1}{2} g\eta^2 \, \mathrm{d}x.$$

They further point out that travelling waves are critical points of the functional $H_{Z,c} := H_Z - c \cdot P$, where

$$P := \int \eta \nabla_x \varphi \, \mathrm{d}x$$

is the momentum. We note that this point of view was used by Craig & Nicholls (2000) in their proof of existence of travelling water waves.

MacKay & Saffman (1986) note that the motion of the eigenvalues associated with an equilibrium of a Hamiltonian, H_c (not necessarily equal to $H_{Z,c}$), are well understood. In particular, if λ is a non-zero, purely imaginary, simple eigenvalue for the problem linearized about the equilibrium, then the second variation of H_c at the equilibrium is either positive or negative definite; this sign is the Krein signature of λ, and it is conserved in the absence of collision with another eigenvalue. A consequence of conservation of energy applied to eigenvalues of the *same* Krein signature is MacKay & Saffman's result.

THEOREM 2.2. *(*MacKay & Saffman 1986*) If two simple, pure imaginary eigenvalues of the same Krein signature collide at a point other than zero, then they cannot leave the imaginary axis.*

Using the fact that Krein signature is conserved up to collision, MacKay & Saffman (1986) point out that for the use of this theorem, one should compute this signature at $\varepsilon = 0$. Here it is easy to find the energy (to second order) explicitly and one discovers that it is a positive multiple of the difference between the speed of the reference frame and the speed of the *perturbation* in this frame. Thus, the energy (and consequently the signature) is negative if the perturbation moves in the same direction but slower than the frame, and positive otherwise.

3. Numerical results

In this section, we outline a numerical method to simulate the recursions (2.11) and then, using the resulting Taylor coefficients $\{v_n, \zeta_n, \lambda_n\}$, deduce information regarding the spectrum as a function of ε. In particular, we show quite explicitly the existence of eigenvalue crossing *before* instability and present new results on singularities of these series. We speculate that spectral instability arises at the smallest (in absolute value) singularity of (2.10) in ε which is further hypothesized to be real valued. We display simulations for four values of the depth, $h=\infty$, $2 > h_c \approx 1.363$, and $h=1, 1/2 < h_c$, where h_c is the critical depth discussed in § 2.7. We find that this distinction will be important in our new criterion of stability, in particular whether eigenvalue collision is sufficient to conclude spectral instability or not.

3.1. *Numerical method*

Our numerical scheme is a Fourier (collocation)/Chebyshev (tau)/Taylor algorithm (see Gottlieb & Orszag 1977; Canuto *et al.* 1988; Nicholls & Reitich 2001*b*, 2006) applied to the system of (2.11). Briefly, this amounts to approximating the unknowns (v, ζ, λ) by

$$v^{(N,N_x,N_y)}(x, y, \varepsilon) := \sum_{n=0}^{N} \sum_{k=-N_x/2}^{N_x/2-1} \sum_{l=0}^{N_y} \hat{v}_n^{k,l} T_l \left(\frac{2y+a}{a} \right) \mathrm{e}^{\mathrm{i}(k+p)x} \varepsilon^n, \tag{3.1a}$$

$$\zeta^{(N,N_x)}(x, \varepsilon) := \sum_{n=0}^{N} \sum_{k=-N_x/2}^{N_x/2-1} \hat{\zeta}_n^k \mathrm{e}^{\mathrm{i}(k+p)x} \varepsilon^n, \tag{3.1b}$$

$$\lambda^{(N)}(\varepsilon) := \sum_{n=0}^{N} \lambda_n \varepsilon^n, \tag{3.1c}$$

where T_l is the l-th Chebyshev polynomial. We determine the $(\hat{v}_n^{k,l}, \hat{\zeta}_n^k, \lambda_n)$ from (2.11) (together with the solvability and uniqueness considerations outlined in Appendix A of Nicholls 2007*b*) and the Fourier collocation (x variable) and Chebyshev tau (y variable) methods. Of course, all of this depends on the faithful computation of the basic travelling wave $(\bar{\varphi}, \bar{\eta}, \bar{c})$ which we perform using the stable high-order TFE method developed in Nicholls & Reitich (2006). In the experiments of this section, we have set $a = 1/2$ ($a=1/10$ for $h=1/2$), $N=30$, $N_x=64$ and $N_y=32$ for the computation of both the travelling waveform $(\bar{\varphi}, \bar{\eta}, \bar{c})$ and the perturbation (v, ζ, λ). These algorithms for the simulation of both the travelling wave and the spectrum of the linearized operator have been thoroughly investigated and verified in the publications (Nicholls & Reitich 2006) and (Nicholls 2007*b*), respectively. In light of these studies, we view our numerical results presented below as accurate and robust.

Before proceeding, we note that there is an alternative method to sum the Taylor series (in ε) appearing in (3.1). It is well known that Padé approximants (see Baker & Graves-Morris 1996) have remarkable properties of approximation for a large class of functions when applied to their Taylor series, not only within their disk of convergence, but also well outside (provided that there are no obstructing singularities present). In a practical implementation, this generally means that a Padé approximation will not only converge *faster* where the Taylor series converges, but it may even converge for values of ε where the Taylor series diverges. We have found this technique useful in a wide range of applications (see Nicholls & Reitich 2003, 2004*b*), and we use it here again. For a full accounting of the advantageous computational complexity of this new scheme, we refer the interested reader to the discussion in Nicholls (2007*b*).

3.2. *Deep water*

We now present a representative sample of new results on the evolution of the spectrum (as a function of ε) of the linearized water-wave operator about fully nonlinear travelling wave solutions of the full water-wave equations (2.5). To begin, we consider the well-studied case of infinitely deep water ($h = \infty$). For this and all future simulations, we consider the case of unit gravity ($g = 1$) and nonlinear Stokes waves periodic of period 2π and basic wavenumber $\bar{k} = 1$, thus $\bar{\eta}_0(x) = \cos(\bar{k}x) = \cos(x)$ and $\bar{c}_0 = 1$ in (2.9).

For these simulations, we studied $128 = 2 \times N_x = 2 \times 64$ ($-32 = -N_x/2 \leqslant k \leqslant N_x/2 - 1 = 31$, and $s = \pm 1$) eigenvalues λ and their corresponding eigenfunctions (ζ, v) with $N_x = 64$ Fourier modes and $N_y = 32$ Chebyshev coefficients. This study was repeated for a sampling of quasi-periods $p \in [0, 1)$ with spacing $\Delta p = 1/100$, excluding the 'resonant' values of $p = 0, 1/4, 3/4$ (see § 2.6). In figure 1, we display a plot of the value of ε where the *first* crossing of eigenvalues occurs for each value of p sampled. This plot is very much in the spirit of the one we produced in Nicholls (2007*b*) save that here we use the Schwartz parameterization rather than Stokes' choice (see § 2.4). Here we notice several things: first, that as $p \to 0$ eigenvalues collide for very small values of ε. These small-amplitude eigenvalue crossings correspond to the Benjamin–Feir instability which results from long-wave modulations and occurs at low steepness. Additionally, at the values $p = 1/4, 3/4$, it appears that eigenvalue collision possibly occurs for small value of ε. Also, crossings for $\varepsilon \ll 1$ occurs as $p \to 1$, but, as the spectral data is periodic with period 1 (see Mielke 1997), this is to be expected. In contrast with these particular values, for all other values of p there is a 'window' of ε where no collision occurs, indicating that even for moderately nonlinear waves there is stability with regard to these quasi-periods.

With this plot in mind, one can wonder if eigenvalue collision *always* gives rise to instability, i.e. does eigenvalue crossing always result in the spectral pair moving into the positive/negative imaginary planes? In figure 2, we provide an answer in the negative: eigenvalue collision does *not* always result in instability. For the case $p = 1/10$, we show in figure 2(*a*) – again, in the spirit of a figure from Nicholls (2007*b*) – the approach of two eigenvalues which begin (at $\varepsilon = 0$) at

$$\lambda_0^1(1) = \mathrm{i}\ \left[-(1)(1 + 1/10) + \omega_{1+1/10}\right] \approx -0.05119\mathrm{i},$$
$$\lambda_0^{-1}(-1) = \mathrm{i}\left[-(1)(-1 + 1/10) - \omega_{-1+1/10}\right] \approx -0.04868\mathrm{i}$$

(see (2.15)). As shown in Nicholls (2007*b*), we can realize a great degree of accuracy (essentially machine precision) in the simulation of these eigenvalues for ε almost to the point of collision. However, as we near this final point, our scheme loses accuracy so that we are only sure of a few digits of accuracy. Significantly, beyond this point we *cannot* continue our calculation reliably as neither Taylor summation nor Padé approximation of (2.10) consistently produces finite values for larger ε. This led us to conjecture that the first collision of eigenvalues and the breakdown of our code occurred simultaneously. However, upon closer investigation of quasi-periods outside of the Benjamin–Feir regime, we discovered that this was *not* necessarily the case. In

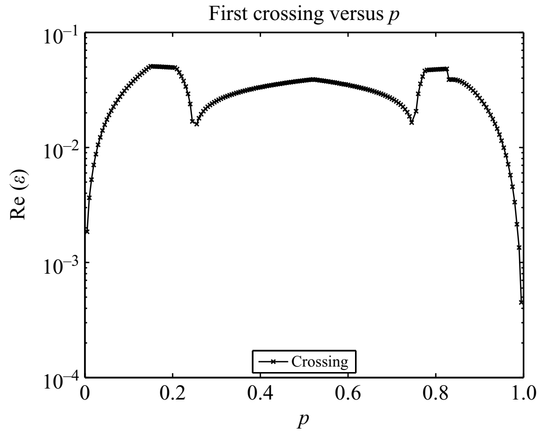

FIGURE 1. Plot of the real part of ε_c, the value of the perturbation parameter where the first eigenvalue crossing occurs, versus p for deep water ($h=\infty$). The underlying Stokes wave is 2π-periodic and Schwartz's parameterization is utilized.

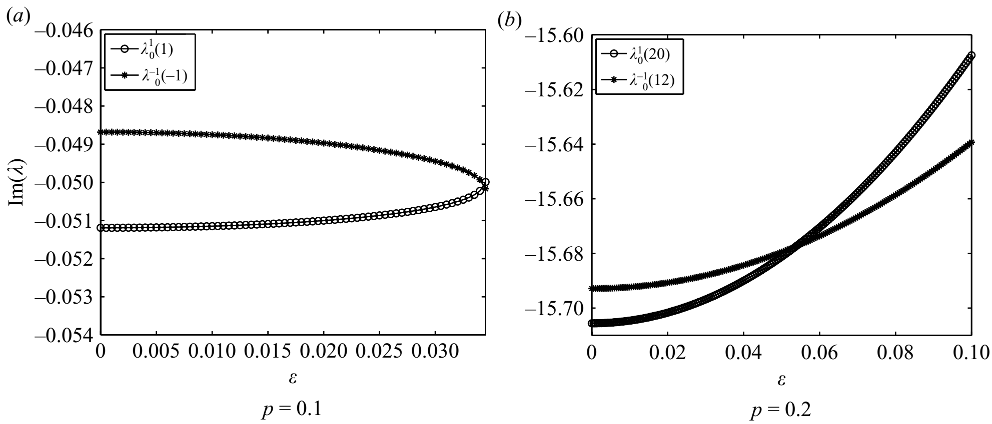

FIGURE 2. Plot of two eigenvalues, $\lambda(\varepsilon)$, versus ε for two different values of the quasi-period ($p=1/10, 2/10$) in water of infinite depth. (*a*) The eigenvalues $\lambda_0^1(1)$ and $\lambda_0^{-1}(-1)$ are plotted and the computation cannot be continued past their collision. (*b*) The eigenvalues $\lambda_0^1(20)$ and $\lambda_0^{-1}(12)$ are displayed, but these computations can be continued *beyond* the collision despite the fact that they have *opposite* Krein signature. The underlying Stokes wave is 2π-periodic and Schwartz's parameterization is utilized.

figure 2(*b*), we display results of a similar calculation with $p=2/10$ and two points of the spectrum which, at $\varepsilon=0$, are (see (2.15))

$$\lambda_0^1(20) = \mathrm{i}\left[-(1)(20+2/10)+\omega_{20+2/10}\right] \approx -15.706\mathrm{i},$$
$$\lambda_0^{-1}(12) = \mathrm{i}\left[-(1)(12+2/10)-\omega_{12+2/10}\right] \approx -15.693\mathrm{i}.$$

In this instance, the two eigenvalues pass over each other and do *not* move off the imaginary axis. Notice also that they do not form a 'bubble of instability', as was noticed for other eigenvalue collisions by Longuet-Higgins (1978*a*, *b*). Clearly, in this instance, eigenvalue crossing did not cause our numerical simulation to break down and did not result in instability; a new criterion is required.

Remark 3.1. *As mentioned earlier, the results of MacKay & Saffman (1986) show that eigenvalue collision is not sufficient for instability. They showed, via the theory of Hamiltonian systems, that this collision must additionally be between eigenvalues of opposite Krein signature (though they did not claim that this is a sufficient condition). For the spectral parameter $\lambda_0^s(\kappa)$, MacKay & Saffman (1986) tell us that the signature is positive if $s\kappa \leqslant -1$, while it is negative if $s\kappa \geqslant 2$. From this, it is easy to compute that $\lambda_0^1(20)$ and $\lambda_0^{-1}(12)$ have opposite signatures (negative and positive, respectively) indicating that this criterion is also insufficient.*

3.3. *A new test for instability: expansion singularities*

The results displayed in figure 2(*b*) indicate that a new test is required for the onset of instability to replace eigenvalue collision. One possibility that we advocate below is inspired, in fact, by the observations of the behaviour of our numerical scheme in the case $p = 1/10$ shown in figure 2(*a*). The failure of our numerical method in this case was due to taking ε 'too large' not only for Taylor summation (meaning that ε is outside the disk of analyticity of our expansion) but, more importantly, also for Padé approximation indicating the presence of a singularity (or singularities) on the real axis. With this in mind, we reflect upon the expansion (2.10), particularly that for $\lambda(\varepsilon)$. On the one hand, we know that $\lambda(0)$ is purely imaginary (implying that λ_0 is pure imaginary) and, furthermore, in the case of a non-resonant configuration, $\lambda(\varepsilon)$ is purely imaginary for ε sufficiently small. This implies that λ_n is purely imaginary for all n, a fact that is borne out (to machine precision) in our numerical simulations. On the other hand, at the onset of instability $\mathrm{Re}\{\lambda(\varepsilon)\} > 0$ requiring some (and most likely all) $\mathrm{Re}\{\lambda_n\} > 0$, a seeming contradiction. The only resolution of this disparity is that the expansion for $\lambda(\varepsilon)$, (2.10) *must* fail when instability arises. Of course, the expansion may fail before the onset of instability, but this seems highly improbable and thus we conjecture.

Conjecture 3.2. *In the absence of resonance (i.e. $p \in \Omega_\kappa(\Gamma)$), the following are equivalent:*

(*a*) *There is a value $\varepsilon_i \in \boldsymbol{R}$ for which*

$$Re\{\lambda^s(\varepsilon;\kappa)\} = 0, \quad \forall \varepsilon < \varepsilon_i, \kappa \in \Gamma',$$

and

$$Re\{\lambda^s(\varepsilon;\kappa)\} > 0, \quad \text{for some } \varepsilon > \varepsilon_i, \text{and some } \kappa \in \Gamma'.$$

This implies spectral instability for $\varepsilon > \varepsilon_i$.

(*b*) *There is a singularity of $\lambda(\varepsilon)$ at $\varepsilon = \varepsilon_i \in \boldsymbol{R}$. This implies divergence of our series (2.10) for $\varepsilon > \varepsilon_i$.*

Remark 3.3. *To state the conjecture more clearly, when an eigenvalue pair leaves the imaginary axis, the expansion for $\lambda(\varepsilon)$, (2.10), must fail since $Re\{\lambda_n\} = 0$. We conjecture that the converse is also true, that failure of (2.10) implies eigenvalue pairs leaving the imaginary axis, and thus spectral instability.*

We point out that this conjecture does *not* exclude the possibility that for a particular p, $\mathrm{Re}\{\lambda(\varepsilon)\} = 0$ for all $\varepsilon \geqslant 0$, i.e. spectral stability for all travelling waves with respect to all disturbances of a given quasi-period. This conjecture permits the behaviour displayed in figure 2(*b*) while agreeing with the intuition gained by noting that $\mathrm{Re}\{\lambda_n\} = 0$ for all n in the absence of resonance.

In light of this conjecture, we display in figure 3 the values of ε of the smallest singularity (which we always found to be real) of the expansion of $\lambda^{(N)}$, (3.1*c*), versus

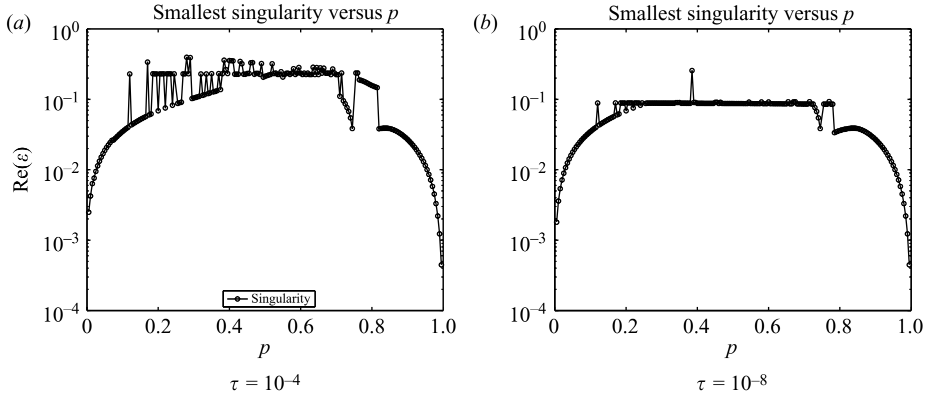

FIGURE 3. Plot of the real part of ε_p, the value of the perturbation parameter where the first singularity in the expansion of $\lambda(\varepsilon)$ occurs, versus p for deep water ($h = \infty$); the imaginary part of ε_p was always found to be zero to machine precision. The underlying Stokes wave is 2π-periodic and Schwartz's parameterization is utilized.

p. In figure 4, we have superimposed the first crossing values for comparison. We have included two plots apiece which vary the singularity cancellation parameter τ from 10^{-4} to 10^{-8}. To explain τ, we note that our method for locating singularities is, given the Padé approximation

$$\lambda^{(N)}(\varepsilon) \approx \frac{p^{(N/2)}(\varepsilon)}{q^{(N/2)}(\varepsilon)} = \frac{\sum_{n=0}^{N/2} a_n \varepsilon^n}{\sum_{n=0}^{N/2} b_n \varepsilon^n},$$

to find the roots of $p^{(N/2)}$ and $q^{(N/2)}$

$$Z_N := \{\varepsilon \in \mathbf{C} \mid p^{(N/2)}(\varepsilon) = 0\}, \quad P_N := \{\varepsilon \in \mathbf{C} \mid q^{(N/2)}(\varepsilon) = 0\},$$

and choose among the ε in

$$S_N(\tau) := \{\varepsilon \in P_N \mid |\varepsilon - \varepsilon_z| > \tau, \quad \forall \varepsilon_z \in Z_N\},$$

which is the set of denominator zeros *not* cancelled by numerator zeros to tolerance τ. We point out that by utilizing this tolerance parameter τ, we can exclude spurious zero/singularity pairs which either should exactly cancel in infinite precision or arise due to numerical noise. For a complete discussion of the efficacy of this approach in locating singularities of functions, we refer the reader to § 2.2 of Baker & Graves-Morris (1996).

We point out that figure 4(a) indicates that first eigenvalue crossing and expansion singularities are synonymous for p near zero and one. However, there are sizeable gaps for other values of p, most notably near $p = 1/4, 3/4$. We interpret this to mean that near $p = 0$ (the Benjamin–Feir regime) infinitesimal waves are unstable. By contrast, it would appear that the resonant configurations $p = 1/4, 3/4$ do *not* give rise to spectral instability from $\varepsilon = 0$, but rather evolution of the spectrum on the imaginary axis for a certain *non-zero* range of ε. A definitive conclusion on this, however, is beyond the current scope of our formulation and we leave this for future work.

REMARK 3.4. *Before leaving these deep water results, we draw a comparison with the work of Longuet-Higgins, in particular the results on sub-harmonic stability in*

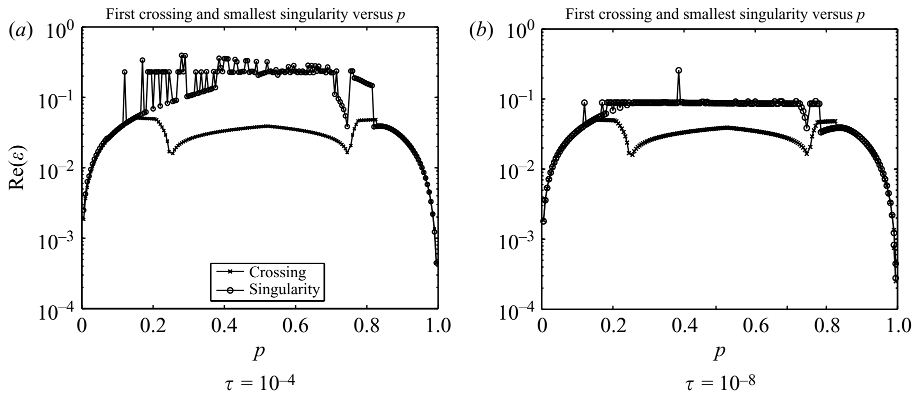

FIGURE 4. Plot of the real parts of ε_c (the value of the perturbation parameter where the first eigenvalue crossing occurs) and ε_p (the value of the perturbation parameter where the first singularity in the expansion of $\lambda(\varepsilon)$ occurs) versus p for deep water ($h=\infty$); the imaginary parts of ε_c and ε_p were always found to be zero to machine precision. The underlying Stokes wave is 2π-periodic and Schwartz's parameterization is utilized.

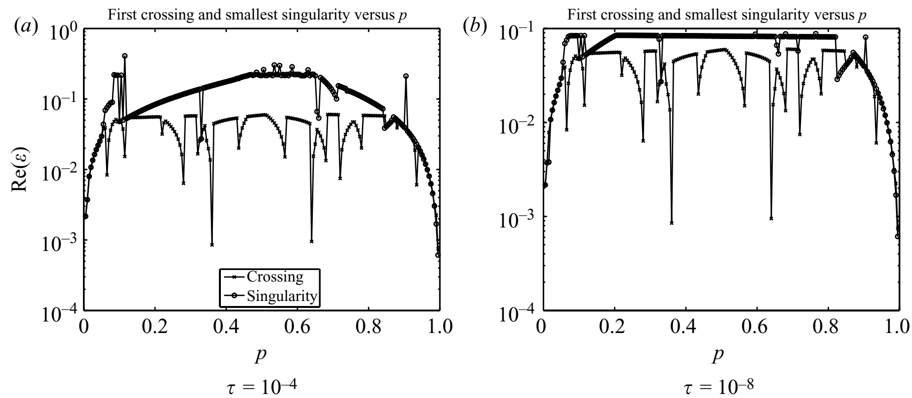

FIGURE 5. Plot of the real parts of ε_c (the value of the perturbation parameter where the first eigenvalue crossing occurs) and ε_p (the value of the perturbation parameter where the first singularity in the expansion of $\lambda(\varepsilon)$ occurs) versus p for water of depth $h=2$; the imaginary parts of ε_c and ε_p were always found to be zero to machine precision. The underlying Stokes wave is 2π-periodic and Schwartz's parameterization is utilized.

Longuet-Higgins (1978b). In this work, a small selection of spectra (between 4 and 16 eigenvalues) is followed in the cases $p=1/2$ *(figures 1 & 2),* $p=1/4$ *(figure 4) and* $p=1/8$ *(figures 5 & 6). Longuet-Higgins (1978b) shows the onset of instability at* $ak\approx 0.22$ *for* $p=1/2$, $ak\approx 0.11$ *for* $p=1/4$ *and* $ak\approx 0.050$ *for* $p=1/8$. *This compares with the results we report in figure 3(b) (with* $\tau=10^{-8}$*) of* $\varepsilon\approx 0.087$ *for* $p=0.5$, $\varepsilon\approx 0.088$ *for* $p=0.255$ *(recall that* $p=1/4$ *is excluded due to resonance) and* $\varepsilon\approx 0.043$ *for* $p=0.125$. *The apparent disparity between the results can be explained by realizing that, since his study considers far fewer eigenvalues, Longuet-Higgins' results can only constitute an upper bound for our results. Of course, our numerics provide only an upper bound for the true set of singular values; however, as our spectrum is far more complete, it is necessarily more accurate.*

For example, while Longuet-Higgins only presents results for four eigenvalues in his study of the $p=1/2$ *case, our value reflects the singularity structure of 128 eigenvalues.*

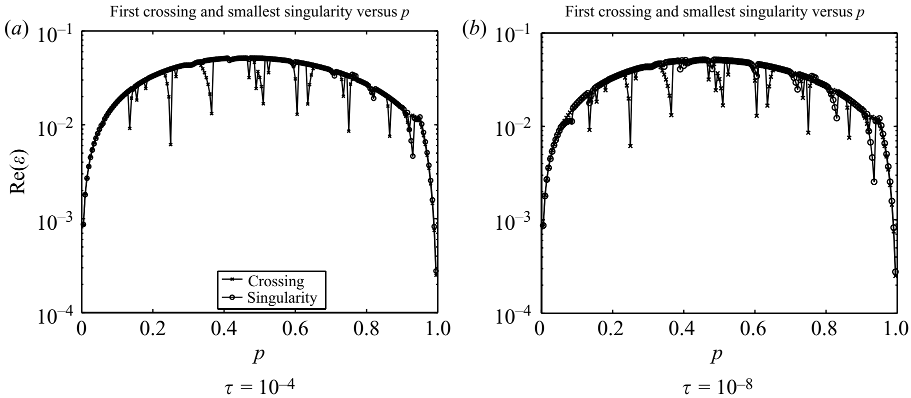

FIGURE 6. Plot of the real parts of ε_c (the value of the perturbation parameter where the first eigenvalue crossing occurs) and ε_p (the value of the perturbation parameter where the first singularity in the expansion of $\lambda(\varepsilon)$ occurs) versus p for water of depth $h=1$; the imaginary parts of ε_c and ε_p were always found to be zero to machine precision. The underlying Stokes wave is 2π-periodic and Schwartz's parameterization is utilized.

In fact, our study of the eigenvalues $\lambda^1(0)$ and $\lambda^{-1}(1)$, corresponding to $n=1/2$ and $n=3/2$, respectively in figure 1 of (Longuet-Higgins 1978b), are qualitatively the same for values of ε up to the first singularity value of $\varepsilon \approx 0.087$. By contrast, in the case $p=1/8$, Longuet-Higgins shows 16 eigenvalues and his results match ours much more closely.

REMARK 3.5. *Returning once again to the notion of a 'bubble of instability', since this is a phenomenon observed subsequent to the onset of instability (i.e. after a collision of eigenvalues which results in their leaving the imaginary axis), our algorithm can find the 'beginning' of such a bubble but cannot study it further. In fact, our figures probably do show the onset of bubbles of instability; however, a different tool (e.g. the method of Longuet-Higgins) is needed to confirm this.*

3.4. *Finite depth*

In this section, we extend the calculations of first eigenvalue crossing and smallest expansion singularity to three other representative depths $h=2, 1, 1/2$. The first, as with the previous value $h=\infty$, is deeper than Benjamin's critical value $h_c \approx 1.363$, while the latter two are shallower and, as we shall see, there is radically different behaviour in this regime. We note, before beginning, the increasing challenge of computing travelling wave solutions as the depth is decreased (moving towards the solitary wave regime); however, we feel confident that our numerics are sufficiently well resolved to draw meaningful conclusions.

For the case $h=2$, we present in figure 5 our results for eigenvalue crossing and smallest singularity as a function of the Bloch period p. Here, we explicitly excluded the value $p=0$, but included all the other values $p=j/100$ for $j=1,\ldots,99$. We see once again that the first eigenvalue crossing and smallest expansion singularities appear, for smallest ε, in the limit $p \to 0$ indicating that the Benjamin–Feir long-wave instability is dominant. Additionally, we see that away from $p=0$, 1, there are regions of ε beyond the first crossing where the first singularity in the expansion has yet to be reached (e.g. for $0.15 < p < 0.8$ in figure 5). This 'window of stability' is similar to the one we found for deep water in the previous section.

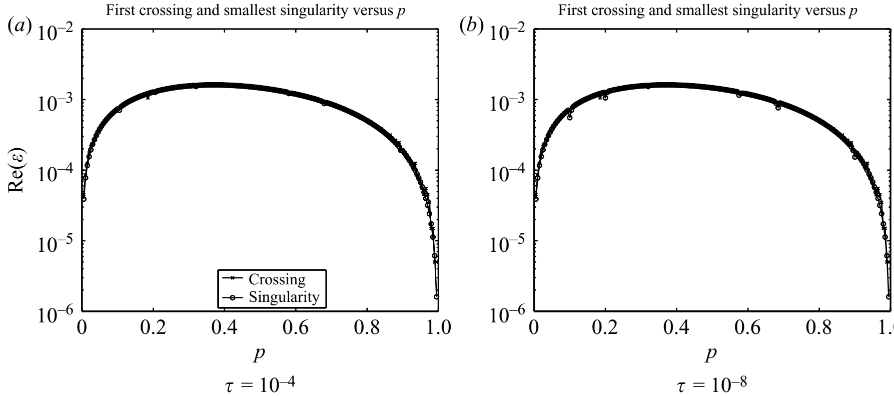

FIGURE 7. Plot of the real parts of ε_c (the value of the perturbation parameter where the first eigenvalue crossing occurs) and ε_p (the value of the perturbation parameter where the first singularity in the expansion of $\lambda(\varepsilon)$ occurs) versus p for water of depth $h = 1/2$; the imaginary parts of ε_c and ε_p were always found to be zero to machine precision. The underlying Stokes wave is 2π-periodic and Schwartz's parameterization is utilized.

In figures 6 and 7, we display results for first crossing and smallest singularity versus quasi-period, p, in the cases $h = 1, 1/2$. We contrast these with the results for deeper water given in figures 4 and 5. On the one hand, it once again seems that $p \to 0$ gives the dominant instability, i.e. instability for the smallest value of ε. While this may no longer be identifiable with the Benjamin–Feir instability, it does seem that Stokes waves are most unstable to long-waves among two-dimensional disturbances. On the other hand, the 'window of stability' disappears in these shallow-water results as first crossing and smallest expansion singularity appear at the same value of the perturbation parameter ε. This is in marked contrast with the deep-water case and presents another novel finding of our computational approach.

REMARK 3.6. *Before closing, we comment on the physical interpretation of the results just presented. The key consideration is the meaning of the quantity ε which, as stated in § 2.4, is the waveheight parameterization of Schwartz (1974). Thus, each of the figures presented above can be used directly to decide if a travelling wave of a particular waveheight (which gives the degree of nonlinearity) is stable or unstable. Simply stated, once waves become sufficiently steep (nonlinear), they are unstable.*

4. Conclusions

We have used a new viewpoint on the spectral stability problem (following spectral data as they vary with a perturbation parameter) to study the onset of instability in Stokes waves as their height is increased. Our study accommodates both super- and sub-harmonic disturbances and ranges over a sampling of four fluid depths, two above and two below the critical depth identified by Benjamin (1967) for the onset of the Benjamin–Feir instability. We have demonstrated with explicit calculations (in the deep water regime) the inadequacy of 'eigenvalue collision' as a diagnostic for determining instability, even collision of opposite Krein signature. In its place, we have conjectured that the 'smallest singularity' in the expansion of the spectral datum $\lambda(\varepsilon)$ might be more appropriate. Using this new measure, we have studied the instabilities of Stokes waves to two-dimensional perturbations and found that long-wave disturbances are dominant. However, we have also discovered a fundamental difference in the onset of instability for shallow waves compared to deep ones (as compared to Benjamin's critical depth $h_c \approx 1.363$): for shallow water eigenvalue collision and smallest eigenvalue singularity are one and the same, while for deep water there are wide 'windows of stability' beyond the first eigenvalue crossing before the first expansion singularity is reached.

The author gratefully acknowledges support from the NSF through grant No. DMS-0537511.

REFERENCES

Baker, Jr., G. A. & Graves-Morris, P. 1996 *Padé Approximants*, 2nd ed. Cambridge University Press.

Benjamin, T. B. 1967 Instability of periodic wavetrains in nonlinear dispersive systems. *Proc. R. Soc. Lond.* A **299** (1456), 59–76.

Benjamin, T. B. & Feir, J. E. 1967 The disintegration of wave trains on deep water. Part I. Theory. *J. Fluid Mech.* **27**, 417–430.

Bridges, T. J. & Mielke, A. 1995 A proof of the Benjamin–Feir instability. *Arch. Rational Mech. Anal.* **133** (2), 145–198.

Canuto, C., Hussaini, M. Y., Quarteroni, A. & Zang, T. A. 1988 *Spectral Methods in Fluid Dynamics*. Springer.

Chandezon, J., Maystre, D. & Raoult, G. 1980 A new theoretical method for diffraction gratings and its numerical application. *J. Opt.* **11** (7), 235–241.

Craig, W. & Nicholls, D. P. 2000 Traveling two and three dimensional capillary–gravity water waves. *SIAM J. Math. Anal.* **32** (2), 323–359.

Deconinck, B. & Kutz, J. N. 2006 Computing spectra of linear operators using the Floquet–Fourier–Hill method. *J. Comput. Phys.* **219** (1), 296–321.

Dias, F. & Kharif, C. 1999 Nonlinear gravity and capillary–gravity waves. In *Annual Review of Fluid Mech*, vol. 31, pp. 301–346. Annual Reviews.

Francius, M. & Kharif, C. 2003 On the disappearance of the lowest-order instability for steep gravity waves in finite depth. *Phys. Fluids* **15** (8), 2445–2448.

Francius, M. & Kharif, C. 2006 Three-dimensional instabilities of periodic gravity waves in shallow water. *J. Fluid Mech.* **561**, 417–437.

Gottlieb, D. & Orszag, S. A. 1977 *Numerical Analysis of Spectral Methods: Theory and Applications*. Society for Industrial and Applied Mathematics, CBMS-NSF Regional Conference Series in Applied Mathematics, No. 26.

Groves, M. D. 2007 Three-dimensional travelling gravity–capillary water waves. *GAMM-Mitt.* **30** (1), 8–43.

Kharif, C. 1987 A comparison between 3 and 2 dimensional instabilities of very steep gravity-waves. *J. Mech. Theor. Appl.* **6** (6), 843–864.

Kharif, C. & Ramamonjiarisoa, A. 1988 Deep-water gravity–wave instabilities at large steepness. *Phys. of Fluids* **31** (5), 1286–1288.

Lamb, H. 1993 *Hydrodynamics*, 6th ed. Cambridge University Press.

Levi-Civita, T. 1925 Détermination rigoureuse des ondes permanentes d'ampleur finie. *Math. Ann.* **93**, 264–314.

Li, L., Chandezon, J., Granet, G. & Plumey, J. P. 1999 Rigorous and efficient grating-analysis method made easy for optical engineers. *Appl. Opt.* **38** (2), 304–313.

Longuet-Higgins, M. S. 1978*a* The instabilities of gravity waves of finite amplitude in deep water. I. Superharmonics. *Proc. R. Soc. Lond. Ser.* A **360** (1703), 471–488.

Longuet-Higgins, M. S. 1978*b* The instabilities of gravity waves of finite amplitude in deep water. II. Subharmonics. *Proc. R. Soc. Lond. Ser.* A **360** (1703), 489–505.

MacKay, R. S. & Saffman, P. G. 1986 Stability of water waves. *Proc. R. Soc. Lond.* A **406**, 115–125.

Marchant, T. R. & Roberts, A. J. 1987 Properties of short-crested waves in water of finite depth. *J. Austral. Math. Soc. Ser.* B **29** (1), 103–125.

McLean, J. W. 1982*a* Instabilities of finite-amplitude gravity waves on water of finite depth. *J. Fluid Mech.* **114**, 331–341.

McLean, J. W. 1982*b* Instabilities of finite-amplitude water waves. *J. Fluid Mech.* **114**, 315–330.

McLean, J. W., Ma, Y. C., Martin, D. U., Saffman, P. G. & Yuen, H. C. 1981 Three-dimensional instability of finite-amplitude water waves. *Phys. Rev. Lett.* **46** (13), 817–820.

Mielke, A. 1997 Instability and stability of rolls in the Swift–Hohenberg equation. *Comm. Math. Phys.* **189** (3), 829–853.

Nicholls, D. P. 2007*a* Boundary perturbation methods for water waves. *GAMM–Mitt.* **30** (1), 44–74.

Nicholls, D. P. 2007*b* Spectral stability of traveling water waves: analytic dependence of the spectrum. *J. Nonlin. Sci.* **17** (4), 369–397.

Nicholls, D. P. & Reitich, F. 2001*a* A new approach to analyticity of Dirichlet–Neumann operators. *Proc. R. Soc. Edinburgh Sect.* A **131** (6), 1411–1433.

Nicholls, D. P. & Reitich, F. 2001*b* Stability of high-order perturbative methods for the computation of Dirichlet–Neumann operators. *J. Comput. Phys.* **170** (1), 276–298.

Nicholls, D. P. & Reitich, F. 2003 Analytic continuation of Dirichlet–Neumann operators. *Numer. Math.* **94** (1), 107–146.

Nicholls, D. P. & Reitich, F. 2004*a* Shape deformations in rough surface scattering: cancellations, conditioning, and convergence. *J. Opt. Soc. Am.* A **21** (4), 590–605.

Nicholls, D. P. & Reitich, F. 2004*b* Shape deformations in rough surface scattering: improved algorithms. *J. Opt. Soc. Am.* A **21** (4), 606–621.

Nicholls, D. P. & Reitich, F. 2005 On analyticity of traveling water waves. *Proc. R. Soc. Lond.* A **461** (2057), 1283–1309.

Nicholls, D. P. & Reitich, F. 2006 Rapid, stable, high-order computation of traveling water waves in three dimensions. *Eur. J. Mech. B Fluids* **25** (4), 406–424.

Phillips, N. A. 1957 A coordinate system having some special advantages for numerical forecasting. *J. Atmosph. Sci.* **14** (2), 184–185.

Reed, M. & Simon, B. 1978 *Methods of Modern Mathematical Physics. IV. Analysis of Operators*. Academic Press (Harcourt Brace Jovanovich).

Roberts, A. J. 1983 Highly nonlinear short-crested water waves. *J. Fluid Mech.* **135**, 301–321.

Schwartz, L. W. 1974 Computer extension and analytic continuation of Stokes' expansion for gravity waves. *J. Fluid Mech.* **62**, 553–578.

Struik, D. 1926 Détermination rigoureuse des ondes irrotationnelles périodiques dans un canal à profondeur finie. *Math. Ann.* **95**, 595–634.

Zakharov, V. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* **9**, 190–194.